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# Polynomial Splines on the Real Line*

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## INTRODUCTION

The splines to be investigated are defined on $(-\infty, \infty)$ with uniformly spaced nodes at the points $x_j = jh$ $(j = 0, \pm 1,...)$, $h$ being a positive number. These splines have been considered recently, [1], [2], [3], particularly for the case $h = 1$. A number of years earlier, they were also investigated in [4]. In [2], [3], existence as well as a number of optimal properties with respect to various norms are established. The present paper proceeds, however, along the lines of [1] and extends the results obtained there for cubic splines to polynomial splines of odd degree. Convergence properties, as $h$ tends to zero, are also investigated.

If the spline in question is of degree $n$, where $n$ is odd, then with relatively mild restrictions on a set of real numbers $\{f_j\}(j = 0, \pm 1,...)$, it is shown that there is a unique spline, $S_h(x)$, with the interpolation property

$$S_h(k_j) = f_j \qquad (j = 0, \pm 1,...) \tag{1}$$

whose $(n - 1)$th derivative is bounded. It is essentially this result that is utilized to establish the indicated convergence properties. Moreover, certain "damping" properties are established that point up the "local" character of spline approximations and are also useful in analyzing convergence. Due to the absence of diagonal dominance in the significant matrix, the methods of [1] do not generalize. The circulant nature of the matrix permits, however, an application of the theory of doubly-infinite Toeplitz matrices to fill this gap.

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## FUNDAMENTAL EQUATIONS

If $S_h(x)$ is a spline of the type under investigation, of degree $n$, where $n$ is odd, let $\bar{n} = (n - 1)/2$. Then, as in [5], we can establish that

$$n!\,(n-1)!\,S_h[x_{i-\bar{n}}, \dots, x_{i+\bar{n}}] = \sum_{j=-\bar{n}}^{\bar{n}} C_j(n)\,S_h^{(n-1)}(x_{i+j}), \tag{2}$$

where the coefficients $C_j(n)$ are the same as those arising in the periodic case for a spline of the same degree and uniformly-spaced nodes. We shall later consider these coefficients in greater detail. Let $M_i = S_h^{(n-1)}(x_i)$. Then for $x_{j-1} \leqslant x \leqslant x_j$ we have

$$S_h^{(n-1)}(x) = M_j\,\frac{(x - x_{j-1})}{h} + M_{j-1}\,\frac{(x_j - x)}{h}, \tag{3}$$

$$S_h^{(n)}(x) = \frac{M_j - M_{j-1}}{h}. \tag{4}$$

Now, let

$$Q_0(x) = \frac{x^n}{n!}\,\frac{M_1 - M_0}{h}. \tag{5}$$

Then there exists a polynomial, $P_{n-1}(x)$, of degree $n - 1$ such that

$$S_h(x) = Q_0(x) + P_{n-1}(x), \qquad x_0 \leqslant x \leqslant x_1. \tag{6}$$

Furthermore, if

$$Q_j(x) = \frac{(x - x_j)^n}{n!}\left[\frac{M_{j+1} - M_j}{h} - \frac{M_j - M_{j-1}}{h}\right], \tag{7}$$

then, for $j > 0$,

$$S_h(x) = Q_0(x) + Q_1(x) + \cdots + Q_j(x) + P_{n-1}(x), \qquad x_j \leqslant x \leqslant x_{j+1}. \tag{7}$$

In a similar fashion we can represent $S_h(x)$ for $x < 0$.

Suppose that $f_j$ ($j = 0, \pm 1, \dots$) are given and for any even integer $\beta$ let $\Delta^\beta f_i$ be the $\beta$-th central difference formed from the $f_j$ and centered at $f_i$. In (2) replace $(n-1)!\,S_h[x_{i-\bar{n}}, \dots, x_{i+\bar{n}}]$ by $(\Delta^{n-1} f_i)/h^{n-1}$ with the result

$$h^{-n+1} n!\,\Delta^{n-1} f_i = \sum_{j=-\bar{n}}^{\bar{n}} C_j(n)\,M_{i+j}. \tag{2'}$$

Also assume for the moment that we can solve the resulting system of equations for $M_j (j = 0, \pm 1, \ldots)$. Using these quantities, form $Q_0(x)$, $Q_1(x)$,... and their counterparts for $x < 0$. Finally, determine the coefficients of $P_{n-1}(x)$ so that $S_h(x)$ will interpolate to $f_j$ at $x_j (j = 0, 1, \ldots, n-1)$. If, for every integer $j$, we let $\hat{f}_j = S_h(x_j)$, then $\hat{f}_j = f_j (j = 0, 1, \ldots, n-1)$. To show that $\hat{f}_j = f_j$ for every $j$, observe that the quantities $M_j$ satisfy equations (2′), with $\Delta^{n-1}\hat{f}_i$ in place of $\Delta^{n-1}f_i$. It follows that there exists a polynomial $P(x)$ of degree $n-2$ such that

$$\hat{f}_j - f_j = P(x_j) \qquad (j = 0, \pm 1, \ldots). \tag{8}$$

However, since $P(x_j) = 0$ for $j = 0, 1, \ldots, n-1$, it follows that $P(x)$ is identically zero. Thus, $\hat{f}_j = f_j$ for every $j$. Consequently, if for a given set $f_j (j = 0, \pm 1, \ldots)$ the quantities $M_j$ are determinable, then $S_h(x)$ is uniquely determined.

We must now show, however, that the last hypothesis is actually fulfilled, i.e., if (for instance) the quantities $h^{-n+1}\Delta^{n-1}f_j$ are bounded they uniquely determine the quantities $M_j$. We shall do more than this; indeed we shall show that the doubly-infinite circulant matrix

$$C(n) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots C_{-1}(n) & C_0(n) & C_1(n) & C_2(n) & C_3(n) \cdots \\ \cdots C_{-2}(n) & C_{-1}(n) & C_0(n) & C_1(n) & C_2(n) \cdots \\ \cdots C_{-3}(n) & C_{-2}(n) & C_{-1}(n) & C_0(n) & C_1(n) \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{9}$$

has an inverse which is bounded in the row-max norm.

*Hille Polynomials.* Hille, [6], introduced a family of polynomials, $P_n(z, a)$, while investigating holomorphy-preserving transformations. The special case where $a = 1$ has already proved useful in spline theory, [7]; we shall make further use of them presently.

For $a = 1$, these polynomials are defined by

$$P_n(z, 1) = z \sum_{k=0}^{n-1} a_{nk} z^k, \tag{10}$$

where the coefficients $a_{nk}$ satisfy the recursion relations

$$\begin{aligned} a_{n0} &= a_{n-1,0} = \cdots = a_{10} = 1, \\ a_{nk} &= (k+1)\, a_{n-1,k} + (n-k)\, a_{n-1,k-1} \qquad (k = 1, 2, \ldots, n-2), \\ a_{n,n-1} &= a_{n-1,n-2}. \end{aligned} \tag{11}$$

It follows from these relations that

$$a_{nk} = a_{n,n-k-1} \,. \tag{12}$$

In tabular form, we have for the initial part of the array:

TABLE 1

Spline Coefficients

| $n$ \ $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 1 | | | | | |
| 3 | 1 | 4 | 1 | | | | |
| 4 | 1 | 11 | 11 | 1 | | | |
| 5 | 1 | 26 | 66 | 26 | 1 | | |
| 6 | 1 | 57 | 302 | 302 | 57 | 1 | |
| 7 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 |

As a consequence of (12), the zeros of $P_n(z, 1)$ occur in reciprocal pairs, with the exception of 0 and $-1$. In [7] they are shown to be distinct, real and non-positive. Moreover, for $n$ odd, $-1$ is not a zero, but, for even $n$, it is. These facts are sufficient for our purposes. It remains to relate these polynomials of Hille to our current investigation of polynomial splines of odd degree. The relation is that, for $n$ odd,

$$\begin{aligned} C_k(n) &= a_{n,\bar{n}+k} \quad (|\,k\,| \leqslant \bar{n}), \\ C_k(n) &= 0 \qquad\quad (|\,k\,| > \bar{n}). \end{aligned} \tag{13}$$

## Toeplitz Matrices

The circulant matrix $C(n)$ in (9) is known as a doubly-infinite Toeplitz matrix. In general, if

$$\phi(\theta) = \sum_{k=-\infty}^{\infty} C_k e^{ik\theta} \qquad (i = \sqrt{-1}), \tag{14}$$

then the matrix

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots C_{-1} & C_0 & C_1 & C_2 & C_3 \cdots \\ \cdots C_{-2} & C_{-1} & C_0 & C_1 & C_2 \cdots \\ \cdots C_{-3} & C_{-2} & C_{-1} & C_0 & C_1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

is denoted by $T_\phi$. Thus, $T_\phi$ is completely determined by the Fourier coefficients of $\phi$ and vice versa. The following theorem can be found in [8] and is essential to our analysis.

THEOREM. *A necessary and sufficient condition that $T_\phi$ be invertible is that $1/\phi$ be essentially bounded. In this case we have*

$$T_\phi^{-1} = T_{1/\phi} . \tag{15}$$

In order to apply this theorem to the problem at hand, we substitute $z = e^{i\theta}$ in

$$\omega(z) = z^{-\bar{n}-1} P_n(z, 1) \tag{16}$$

and denote the result by $Q_n(\theta)$. Thus, in view of (13),

$$Q_n(\theta) = \sum_{k=-\infty}^{\infty} C_k(n)\, e^{ik\theta}. \tag{17}$$

Consequently, since $P_n(z, 1)$ has no zeros on the unit circle for $n$ odd, we have

$$Q_n(\theta) \neq 0 \qquad -\pi \leqslant \theta \leqslant \pi. \tag{18}$$

It follows that $T_{Q_n}^{-1}$ exists and the coefficients $a_k$ of this inverse matrix are given by

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{(\bar{n}+1)i\theta} e^{ik\theta}}{P_n(\theta)}\, d\theta \qquad (k = 0, \pm 1, \ldots), \tag{19}$$

where $P_n(\theta) = P_n(e^{i\theta}, 1)$. These arguments establish the following

THEOREM 1. *The matrix $C(n)$ defined by* (9) *has an inverse whose defining coefficients $a_k (k = 0, \pm 1, \ldots)$ are square summable.*

This theorem is not, however, entirely satisfactory for our purposes. We would like to have not only

$$\sum_{k=-\infty}^{\infty} a_k^2 < \infty, \tag{20}$$

but also

$$\sum_{k=-\infty}^{\infty} | a_k | < \infty, \tag{21}$$

i.e., that the inverse matrix $T_{Q_n}^{-1}$ is bounded in the row-max norm. We now focus our attention on this problem.

## DAMPING PROPERTIES

As we have seen, $T_{Q_n}^{-1}$, as a linear operator, is bounded in the Hilbert space sense. In order to see that it is bounded with respect to the row-max norm, we need the following well-known result.

THEOREM. *If $\phi(\theta)$ and its first $p - 1$ derivatives are continuous and the p-th derivative is of bounded variation, the asymptotic behavior of the Fourier coefficients, $A_k$, of $\phi(x)$ is given by*

$$A_k = 0\left(\frac{1}{k^{p+1}}\right), \qquad k \to \infty. \tag{22}$$

*Moreover,*

$$\sum_{|k|>k_0} | A_k | = 0\left(\frac{1}{{k_0}^p}\right), \qquad k_0 \to \infty. \tag{23}$$

To apply this theorem to our problem, observe that $1/\phi(\theta)$, as a function of $\theta$ on $[-\pi, \pi]$, is analytic. Consequently, we have the following

THEOREM 2. *The matrix $T_{Q_n}$ has an inverse which is bounded with respect to the row-max norm. Furthermore, for every positive integer p, we have*

$$a_k = 0\left(\frac{1}{k^{p+1}}\right), \qquad k \to \infty, \tag{24}$$

*and*

$$\sum_{|k|>k_0} | a_k | = 0\left(\frac{1}{{k_0}^p}\right), \qquad k_0 \to \infty. \tag{25}$$

Although the theorem just established gives rather strong decay properties for the coefficients $a_k$ determining $T_{Q_n}^{-1}$, as $k \to \infty$, it is not the best possible result. It turns out that we can determine the coefficients $a_k$ explicitly and make an even stronger assertion concerning their asymptotic behavior. From (19) we have

$$\begin{aligned} a_{-k} &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{ik\theta} e^{\bar{n}i\theta} i e^{i\theta}\, d\theta}{P_n(\theta)} \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{k+\bar{n}}}{P_n(z, 1)}\, dz. \end{aligned} \tag{26}$$

It follows that $a_{-k}$ is the sum of the residues of $(z^{k+\bar{n}}/P_n(z, 1))$ within $| z | = 1$.

Let $x_j$ ($j = 0, 1, \ldots, \bar{n}$) denote the zeros of $P_n(z, 1)$ within $|z| = 1$ and let $0 = x_0 > x_1 > \cdots > -1$. Since the function

$$F(z) = \frac{z^{k+\bar{n}}}{P_n(z, 1)} \tag{27}$$

is the quotient of two holomorphic functions and since $P_n(z, 1)$ has only simple zeros, the residue at $z = x_j$ ($j = 0, 1, \ldots, \bar{n}$) is

$$\frac{x_j^{k+\bar{n}}}{P_n'(x_j, 1)}.$$

Thus, in view of (12) and (13),

$$a_k = a_{-k} = \sum_{j=0}^{\bar{n}} \frac{x_j^{k+\bar{n}}}{P_n'(x_j, 1)} = \sum_{j=1}^{\bar{n}} \frac{x_j^{k+\bar{n}}}{P_n'(x_j, 1)}, \tag{28}$$

and we are led to the following

THEOREM 3. *The coefficients $a_k$ defining $T_{Q_n}^{-1}$ are given by* (28), *and*

$$a_k = 0(|x_{\bar{n}}|^{|k|})$$

as $|k| \to \infty$.

## UNIFORM CONVERGENCE

We have previously established that $C(n)^{-1}$ exists and is bounded with respect to the row-max norm. Let

$$C(n)^{-1} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots a_1(n) & a_0(n) & a_1(n) & a_2(n) & a_3(n) \cdots \\ \cdots a_{-2}(n) & a_{-1}(n) & a_0(n) & a_1(n) & a_2(n) \cdots \\ \cdots a_{-3}(n) & a_{-2}(n) & a_{-1}(n) & a_0(n) & a_1(n) \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{29}$$

Then there exists a constant $M$ such that

$$\sum_{j=-\infty}^{\infty} |a_j(n)| < M. \tag{30}$$

From (29) and (2′) we have

$$M_i = n! \sum_{j=-\infty}^{\infty} a_j(n) \frac{\Delta^{n-1} f_{i+j}}{h^{n-1}}. \tag{31}$$

It now follows that if $f(x)$ has a bounded and continuous $(n-1)$th derivative on $(-\infty, \infty)$ and $f_j = f(jh)$ $(j = 0, \pm 1, \ldots)$, then

$$M_i = n! \sum_{j=-\infty}^{\infty} a_j(n) f^{(n-1)}(\xi_{i+j}), \tag{32}$$

where $x_{j-\bar{n}} \leqslant \xi_j - x_{j+\bar{n}}$. Moreover, due to the circulant nature of (29),

$$\begin{aligned} | M_i - M_k | &\leqslant \sum_{j=-\infty}^{\infty} | a_j(n)| \, | f^{(n-1)}(\xi_{j+i}) - f^{(n-1)}(\xi_{j+k})| \\ &\leqslant M\omega(f^{(n-1)}; (| i - k | + n)h), \end{aligned} \tag{33}$$

where $\omega(g; \delta)$ is the modulus of continuity of $g(x)$. Now, in [6] it is shown that

$$\sum_{j=-\bar{n}} C_j(n) = n!; \tag{34}$$

hence, from (2′) we obtain

$$M_i = \frac{\Delta^{n-1} f_i}{h^{n-1}} + \frac{1}{n!} \sum_{j=-\bar{n}}^{\bar{n}} C_j(n)(M_i - M_{i+j}), \tag{35}$$

so that

$$| M_i - f^{(n-1)}(\xi_i)| \leqslant M\omega(f^{(n-1)}; (\bar{n} + n)\, h). \tag{36}$$

Thus, from (35), the linearity of $S_h^{(n-1)}(x)$, and the triangle inequality, we obtain the following

THEOREM 4. *Let $f(x)$ have a bounded and continuous $(n-1)$th derivative on $(-\infty, \infty)$. Then*

$$S_h^{(n-1)}(x) \to f^{(n-1)}(x) \tag{37}$$

*uniformly, as $h \to 0$, and*

$$| S_h^{(n-1)}(x) - f^{(n-1)}(x)| = 0\{\omega[f^{(n-1)}; \tfrac{3}{2}nh]\}, \tag{38}$$

*where $S_h(x)$ is the spline of interpolation to $f(x)$ at the nodes*

$$x_j = jh (j = 0, \pm 1, \ldots).$$

Using Rolle's theorem and simple quadratures repeatedly, we obtain the

COROLLARY. *Under the conditions of Theorem* 4,

$$S_h^{(\alpha)}(x) - f^{(\alpha)}(x) = 0(h^{n-1-\alpha}\omega(f^{(n-1)}, \tfrac{3}{2}nh)) \tag{39}$$

*for* $\alpha = 0, 1, \ldots, n-1$.

Suppose now that $f^{(n)}(x)$, as well, is bounded and continuous on $(-\infty, \infty)$. Then, letting $N_j = S_h^{(n)}(x)$ for $x_{j-1} < x < x_j$ and using (4), we obtain by differencing (2′) for consecutive values of the index $i$ and dividing by $h$ (we let this process define $\Delta^n f_i$):

$$n!\,\frac{\Delta^n f_i}{h^n} = \sum_{j=-n}^{\bar{n}} C_j(n)\, N_{i+j}\,. \tag{40}$$

Consequently, we can repeat the proof of Theorem 4 virtually unaltered and obtain

THEOREM 5. *Let $f(x)$ have a bounded and continuous n-th derivative on* $(-\infty, \infty)$. *Then*

$$S_h^{(n)}(x) \to f^{(n)}(x) \tag{41}$$

*uniformly, as* $h \to 0$.

We also have

COROLLARY 1. *Under the conditions of Theorem* 5,

$$| S_h^{(\alpha)}(x) - f^{(\alpha)}(x)| = 0(h^{n-\alpha}\omega(f^{(n)}; (\tfrac{3}{2}n + 1)h)) \tag{42}$$

*for* $\alpha = 0, 1, \ldots, n$.

If $f^{(n)}(x)$ is absolutely continuous, we have

$$\omega(f^{(n)}; \delta) = \sup_{|x-y|\leqslant\delta} \left| \int_x^y f^{(n+1)}(t)\, dt \right|, \tag{43}$$

from which the following corollary is evident.

COROLLARY 2. *Under the conditions of Theorem* 5 *and the additional assumption that $f^{(n+1)}(x)$ is bounded and continuous,*

$$S_h^{(\alpha)}(x) - f^{(\alpha)}(-x) = 0(h^{n+1-\alpha}) \tag{44}$$

for $\alpha = 0, 1, \ldots, n$.

## LOCAL CONVERGENCE

We now consider the local behavior of spline approximations. Let

$$\omega_x(g; \delta) = \sup_{\{y \mid |y-x| < \delta\}} | g(x) - g(y)|, \tag{45}$$

so that $\omega_x(g, \delta)$ is a local modulus of continuity for $g(x)$ depending only on the behavior of $g(x)$ in a $\delta$-neighborhood of $x$. Suppose that $f^{(n-1)}(x)$ exists and is bounded on $(-\infty, \infty)$. Then

$$\begin{aligned} S_h^{(n-1)}(x_j) \equiv M_j &= n! \sum_{k=-\infty}^{\infty} a_k(n) \frac{\Delta^{n-1} f_{k+j}}{h^{n-1}} \\ &= n! \sum_{k=-\infty}^{\infty} a_k(n) f^{(n-1)}(\xi_{k+j}) \\ &= n! \sum_{k=-\infty}^{\infty} a_k(n) f^{(n-1)}(x_j) + n! \sum_{k=-\infty}^{\infty} a_k(n)[f_k^{(n-1)}(\xi_{k+j}) - f^{(n-1)}(x_j)], \end{aligned} \tag{46}$$

where $x_k - \bar{n} \leqslant \xi_k \leqslant x_k + \bar{n}$. From (34) it follows that

$$\sum_{k=-\infty}^{\infty} a_k(n) = \frac{1}{n!}; \tag{47}$$

consequently,

$$M_j = f^{(n-1)}(x_j) + n! \sum_{k=-\infty}^{\infty} a_k(n)[f^{(n-1)}(\xi_{k+j}) - f^{(n-1)}(x_j)]. \tag{48}$$

But,

$$\begin{aligned} &\left| \sum_{k=-\infty}^{\infty} a_k(n)[f^{(n-1)}(\xi_{k+j}) - f^{(n-1)}(x_j)] \right| \\ &\quad \leqslant \sum_{k=-N}^{N} | a_k(n)| \, \omega_{x_j}(f^{(n-1)}; (N+n)h) + 2 \| f^{(n-1)} \|_\infty \sum_{|k|>N} | a_k(n)|, \end{aligned} \tag{49}$$

where

$$\| g \|_\infty = \sup_x | g(x)|. \tag{50}$$

From (48), (49) and the linearity of $S_h^{(n-1)}(x)$ between nodes, we are led to the following

THEOREM 6. *Let $f(x)$ have a bounded $(n-1)$-th derivative on $(-\infty, \infty)$ and let $\omega_x(f^{(n-1)}; \delta) \to 0$ as $\delta \to 0$. Then*

$$S_h^{(n-1)}(x) \to f^{(n-1)}(x) \tag{51}$$

*as $h \to 0$, and*

$$| S_h^{(n-1)}(x) - f^{(n-1)}(x)| \leqslant \sum_{k=-N}^{N} | a_k(n)| \, \omega_k(f^{(n-1)}; (n+N)h)$$

$$+ 2 \| f^{(n-1)} \|_\infty \sum_{k>N} | a_k(n)|. \tag{52}$$

Observe that since $| a_k | = 0(r_n^k)$, as $k \to \infty$, the second term converges to zero rapidly as $N \to \infty$, and for a fixed $N$, the second term converges to zero as $h \to 0$. If we let $N > | \ln h |$, then the first assertion follows. We also have the

COROLLARY. *Under the conditions of Theorem 6,*

$$| S_h^{(\alpha)}(x) - f^{(\alpha)}(x)| \leqslant h^{n-1-\alpha} | S_h^{n-1}(x) - f^{(n-1)}(x)| \tag{53}$$

*for $\alpha = 0, 1, \ldots, n-1$.*

As before, if $f^{(n)}(x)$ exists and is bounded on $(-\infty, \infty)$, we obtain the following results.

THEOREM 7. *Let $f(x)$ have a bounded n-th derivative on $(-\infty, \infty)$ and let $\omega_x(f^{(n)}; \delta) \to 0$ as $\delta \to 0$. Then*

$$S_h^{(n)}(x) \to f^{(n)}(x) \tag{54}$$

*as $h \to 0$, and*

$$| S_h^{(n)}(x) - f^{(n)}(x)| \leqslant \sum_{k=-N}^{N} | a_k(n)| \, \omega_x(f^{(n)}; (n+N+1)h)$$

$$+ 2 \| f^{(n)} \|_\infty \sum_{k>N} | a_k(n)|. \tag{55}$$

COROLLARY. *Under the conditions of theorem 7,*

$$| S_h^{(\alpha)}(x) - f^{(\alpha)}(x)| \leqslant h^{n-\alpha} | S_h^{n-1}(x) - f^{(n-1)}(x)| \tag{56}$$

*for $\alpha = 0, 1, \ldots, n$.*

If $f^{(n+1)}$ exists and is bounded in a neighborhood of $x$, we can replace (56) by

$$| S_h^{(\alpha)}(x) - f^{(\alpha)}(x)| \leqslant h^{n+1-\alpha} \sup_{\{y \mid |y-x| < (n+1+N)h\}} | f^{(n+1)}(y)| \sum_{k=-N}^{N} | a_k(n)| + 2h^{n-\alpha} \| f^{(n)} \|_\infty \sum_{k>N} | a_k(n)| \tag{59}$$

for $\alpha = 0, 1, \ldots, n$.

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